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AN ALGORITHM FOR DISCONTINUOUS INVERSE STURM–LIOUVILLE PROBLEMS WITH SYMMETRIC POTENTIALS

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Abstract—In this paper we consider two Sturm–Liouville problems with symmetric potentials and symmetric discontinuities satisfying symmetric boundary and jump conditions. In the first section we derive a simple expression for the difference of potentials when only a finite number of eigenvalues differ. In the second section this result is used to construct an algorithm for solving the discontinuous inverse Sturm–Liouville problem numerically.

1. THE DIFFERENCE OF TWO POTENTIALS

In this section we consider two Sturm–Liouville problems with different symmetric potentials with symmetric discontinuities satisfying different symmetric boundary and jump conditions. The main result is that if only a finite number of eigenvalues differ then a simple expression for the difference of the potentials can be established.

Theorem 1

Consider the following eigenvalue problems with symmetric discontinuities at $x = d_1$ and $x = d_2 = \pi - d_1$:

$$\begin{aligned} -u'' + q(x)u &= \lambda u \\ hu(0) - u'(0) = 0, &\quad hu(\pi) + u'(\pi) = 0 \\ u(d_1+) = au(d_1-), &\quad u'(d_1+) = a^{-1}u'(d_1-) + bu(d_1-) \\ u(d_2-) = au(d_2+), &\quad u'(d_2-) = a^{-1}u'(d_2+) - bu(d_2+) \end{aligned} \tag{1}$$

and symmetric discontinuities at $x = \tilde{d}_1$ and $x = \tilde{d}_2 = \pi - \tilde{d}_1$:

$$\begin{aligned} -u'' + \tilde{q}(x)u &= \tilde{\lambda} u \\ \tilde{h}u(0) - u'(0) = 0, &\quad \tilde{h}u(\pi) + u'(\pi) = 0 \\ u(\tilde{d}_1+) = au(\tilde{d}_1-), &\quad u'(\tilde{d}_1+) = a^{-1}u'(\tilde{d}_1-) + \tilde{b}u(\tilde{d}_1-) \\ u(\tilde{d}_2-) = au(\tilde{d}_2+), &\quad u'(\tilde{d}_2-) = a^{-1}u'(\tilde{d}_2+) - \tilde{b}u(\tilde{d}_2+). \end{aligned} \tag{2}$$

Here q and $\tilde{q}$ are integrable on $(0, \pi)$ and satisfy the symmetry conditions $q(x) = q(\pi - x)$ and $\tilde{q}(x) = \tilde{q}(\pi - x)$ almost everywhere on the interval $0 \leqslant x \leqslant \pi$. The jump constants satisfy $|a - 1| + |b| > 0$. Finally, λ_j and $\tilde{\lambda}_j$ are the eigenvalues of (1) and (2). Let $\tilde{u}_j$ and $\tilde{v}_j$ be the solutions of

$$\begin{aligned} -u'' + \tilde{q}(x)u &= \lambda u \\ u(\tilde{d}_1+) = \tilde{a}u(\tilde{d}_1-), &\quad u'(\tilde{d}_1+) = \tilde{a}^{-1}u'(\tilde{d}_1-) + \tilde{b}u(\tilde{d}_1-) \\ u(\tilde{d}_2-) = \tilde{a}u(\tilde{d}_2+), &\quad u'(\tilde{d}_2-) = \tilde{a}^{-1}u'(\tilde{d}_2+) - \tilde{b}u(\tilde{d}_2+) \end{aligned} \tag{3}$$

$$u(0) = 1, \quad u'(0) = \tilde{h} \tag{4}$$

$$v(\pi) = 1, \quad v'(\pi) = -\tilde{h} \tag{5}$$

with $\lambda = \lambda_j$. Define the functions $\tilde{y}$ by

$$\tilde{y}_j = 2\frac{\tilde{v}_j - k_j\tilde{u}_j}{\omega'(\lambda_j)}. \tag{6}$$

Here

$$k_j/\omega'(\lambda_j) = 1\bigg/\int_0^\pi u_j^2\,\mathrm{d}x$$

where $k_j = (-1)^j$ and $u_j(x)$ are the eigenfunctions of (1) normalized such that $u_j(0) = 1$. If $\lambda_j = \tilde{\lambda}_j$ for $j > n$, then

$$h - \tilde{h} = \frac{1}{2}\sum_{j=0}^{n} \tilde{y}_j(0) \tag{7}$$

$$b - \tilde{b} = \frac{1}{2}(a^3 - a^{-1})\sum_{j=0}^{n} \tilde{y}_j(d_1-)u_j(d_1-) \tag{8}$$

$$q - \tilde{q} = \sum_{j=0}^{n} (\tilde{y}_j u_j)' \text{ a.e.} \tag{9}$$

Remarks. In [1] Hald remarks that a work by Hochstadt [2] inspired him to examine the possibility of constructing an algorithm to solve the continuous symmetric inverse Sturm–Liouville problem numerically. In the continuous problem Hald assumes that an infinite number of eigenvalues could differ so long as the sum

$$\sum_j |\lambda_j - \tilde{\lambda}_j|$$

converges. This always holds for perturbation of finitely many eigenvalues. Hald's result is significant in that it allows him to determine an algorithm to reconstruct the potential function. Under certain assumptions he can prove that the algorithm has a solution and that this solution is unique. Here we extend Hald's characterization results to the discontinuous symmetric inverse Sturm–Liouville problem where a finite number of eigenvalues differ. The techniques of Hald [1] and Hochstadt [2] are used to derive expressions for the difference between the boundary constants and the difference between the potential functions. Since we consider the discontinuous problem, we also obtain a formula for the difference between the jump constants. This is our main contribution. These formulae are used in the second section to construct an algorithm to determine the potential function. The proof below uses results from a uniqueness theorem by Kobayashi [3] and a Sturm–Liouville expansion derived in [3].

Beginning of proof

We begin by noting that $a = \tilde{a}$ and $d = \tilde{d}$ since $\{\lambda_j\}$ are identical for $j > n$ (see Kobayashi [3]). For the remainder of this proof a and d will be used in the place of $\tilde{a}$ and $\tilde{d}$. Let $u(x, \lambda)$ and $v(x, \lambda)$ be the solutions of equation (1), (4)–(5), where $\tilde{h}$ is replaced by h. We let $\tilde{u}$ and $\tilde{v}$ be defined as above in equations (3)–(5). If f and f' are sectionally continuous on $(0, \pi)$ with sections $(0, d)$, $(d, \pi - d)$ and $(\pi - d, \pi)$ then by a corollary in Kobayashi [3] f has an expansion

$$f(x) = \sum_{j=0}^{\infty} \frac{\tilde{v}_j \int_0^x u_j f\,\mathrm{d}y + \tilde{u}_j \int_x^\pi v_j f\,\mathrm{d}y}{\omega'(\lambda_j)} \tag{10}$$

for $0 \leqslant x \leqslant \pi$, $x \neq d_1, d_2$.

We note that u_j and v_j represent the same eigenfunction whereas $\tilde{u}_j$ and $\tilde{v}_j$ are not necessarily eigenfunctions. q is symmetric so that $v_j = k_j u_j$ where $k_j = (-1)^j$. When $q = \tilde{q}$ and $h = \tilde{h}$ then (10) reduces to the Sturm–Liouville expansion and consequently

$$k_j/\omega'(\lambda_j) = 1\bigg/\int_0^\pi u_j^2\,\mathrm{d}x.$$

(see Kobayashi [3]). Let f be the eigenfunction u_0 of (1), and substitute into equation (10). Then

$$u_0 = \tilde{u}_0 + \frac{1}{2}\sum_{j=0}^{n} \tilde{y}_j \int_0^x u_j u_0\,\mathrm{d}t, \tag{11}$$

where we have used that $\tilde{y}_j = 0$ for $j > n$. Formally differentiate equation (11). Let

$$f_j = \tilde{y}_j \int_0^x u_j u_0 \, dt.$$

Then $f_j(0) = 0$ and $f_j'(0) = \tilde{y}_j(0)$. Note that u_j and u_0 are eigenfunctions of (1) and $\tilde{y}_j$ is a solution of (3) with $\lambda = \lambda_j$. We will differentiate f_j twice and use integration by parts to show that

$$f_j'' + (\lambda_0 - \tilde{q})f_j = 2(\tilde{y}_j u_j)' u_0. \tag{12}$$

The expressions for f_j, f_j' and f_j'' are

$$f_j = \tilde{y}_j \int_0^x u_j u_0$$

$$f_j' = \tilde{y}_j u_j u_0 + \tilde{y}_j' \int_0^x u_j u_0$$

$$f_j'' = 2\tilde{y}_j' u_j u_0 + \tilde{y}_j u_j' u_0 + \tilde{y}_j u_j u_0' + \tilde{y}_j'' \int_0^x u_j u_0.$$

Substitute into the left-hand side of equation (12) to find

$$f_j'' + (\lambda_0 - \tilde{q})f_j = 2(\tilde{y}_j u_j)' u_0 + (\lambda_0 - \lambda_j)\tilde{y}_j \int_0^x u_j u_0 + (u_j u_0' - u_j' u_0)\tilde{y}_j. \tag{13}$$

Consider the integral term

$$\begin{aligned} I &= (\lambda_0 - \lambda_j) \int_0^x u_j u_0 \\ &= \int_0^x u_j \lambda_0 u_0 - \int_0^x \lambda_j u_j u_0 \\ &= \int_0^x u_j(\tilde{q} u_0 - u_0'') - \int_0^x (\tilde{q} u_0 - u_0'') u_0 \\ &= \int_0^x (u_j'' u_0 - u_0'' u_j) \\ &= \int_0^x (u_j' u_0 - u_0' u_j)' \end{aligned}$$

which equals

$$(u_j' u_0 - u_0' u_j)|_0^x$$

for $0 < x < d$,

$$(u_j' u_0 - u_0' u_j)|_0^{d-} + (u_j' u_0 - u_0' u_j)|_{d+}^x$$

for $d < x < \pi - d$, and

$$(u_j' u_0 - u_0' u_j)|_0^{d-} + (u_j' u_0 - u_0' u_j)|_{d+}^{(\pi - d)-} + (u_j' u_0 - u_0' u_j)|_{(\pi - d)+}^x$$

for $\pi - d < x < \pi$. In all three cases we use the boundary and jump conditions to show that I equals $(u_j' u_0 - u_0' u_j)(x)$. Since

$$u_0 - \tilde{u}_0 = \frac{1}{2} \sum_{j=0}^{n} f_j,$$

if follows that

$$u_0' - \tilde{u}_0' = \frac{1}{2}\sum_{j=0}^{n} f_j'$$

$$u_0'' - \tilde{u}_0'' = (\tilde{q} - \lambda_0)(u_0 - \tilde{u}_0) + \sum_{j=0}^{n} (\tilde{y}_j u_j)' u_0.$$

To derive equation (7) let $x = 0$ in the first equation. To derive (9) use the second equation, the relations $u_0'' = (q - \lambda_0)u_0$ and $\tilde{u}_0'' = (\tilde{q} - \lambda_0)\tilde{u}_0$ and note that the eigenfunction u_0 is positive in the whole interval. Finally we will use equations (11) and (13) to determine formula (8) for $b - \tilde{b}$. To simplify the notation we will write $(+)$ for (d_1+) and $(-)$ for (d_1-). Let $x = d-$ in formula (11) then

$$u_0(-) = \tilde{u}_0(-) + \frac{1}{2}\sum_{j=0}^{n} \tilde{y}_j(-)\int_0^d u_j u_0.$$

We differentiate (11):

$$u_0'(x) = \tilde{u}_0'(x) + \frac{1}{2}\sum_{j=0}^{n}\left(\tilde{y}_j'(x)\int_0^x u_j u_0 + \tilde{y}_j(x)u_j(x)u_0(x)\right). \tag{14}$$

Let $x = d-$ and let $x = d+$ then

$$u_0'(-) = \tilde{u}_0'(-) + \frac{1}{2}\sum_{j=0}^{n}\left(\tilde{y}_j'(-)\int_0^d u_j u_0 + \tilde{y}_j(-)u_j(-)u_0(-)\right), \tag{15}$$

and

$$u_0'(+) = \tilde{u}_0'(+) + \frac{1}{2}\sum_{j=0}^{n}\left(\tilde{y}_j'(+)\int_0^d u_j u_0 + \tilde{y}_j(+)u_j(+)u_0(+)\right). \tag{16}$$

Next substitute in the jump conditions for u into equation (16) above then

$$\begin{aligned} a^{-1}u_0'(-) + bu_0(-) = a^{-1}\tilde{u}_0'(-) + \tilde{b}\tilde{u}_0(-) &+ \frac{1}{2}\sum_{j=0}^{n}(a^{-1}\tilde{y}_j'(-) + \tilde{b}\tilde{y}_j(-))\int_0^d u_j u_0 \\ &+ \frac{1}{2}\sum_{j=0}^{n} a\cdot\tilde{y}_j(-)\cdot a\cdot u_j(-)\cdot a\cdot u_0(-). \end{aligned}$$

Multiply equation (15) by a^{-1} use the equation to cancel terms in the expression above then

$$\begin{aligned} bu_0(-) &= \tilde{b}\tilde{u}_0(-) + \tilde{b}\frac{1}{2}\sum_{j=0}^{n}\tilde{y}_j'(-)\int_0^d u_j u_0 + \frac{1}{2}\sum_{j=0}^{n}(a^3 - a^{-1})\tilde{y}_j(-)u_j(-)u_0(-) \\ &= \tilde{b}u_0(-) + \frac{1}{2}\sum_{j=0}^{n}(a^3 - a^{-1})\tilde{y}_j(-)u_j(-)u_0(-). \end{aligned}$$

Finally divide by $u_0(-)$:

$$b = \tilde{b} + \tfrac{1}{2}(a^3 - a^{-1})\sum_{j=0}^{n}\tilde{y}_j(-)u_j(-).$$

This completes the proof.

The formulae derived in the theorem above enable us to present a more elegant uniqueness proof for discontinuous symmetric inverse Sturm–Liouville problems. The symmetries of the eigenfunctions and potential functions are fully exploited to give a concise and clear proof.

Corollary 1 (a second uniqueness proof)

Consider the eigenvalue problem (1) where q is integrable on $0 \leqslant x < \pi$. If $q(x) = q(\pi - x)$ almost everywhere in $0 < x < \pi$ and $|a - 1| + |b| > 0$ then $q(x)$, a, b and h are uniquely determined by the spectrum.

Proof. Assume that we have two Sturm–Liouville problems with the same eigenvalues $\lambda_j = \tilde{\lambda}_j$. By a corollary in Kobayashi [3] a equals $\tilde{a}$. From equations (3) and (4) it follows that $\tilde{u}_j$ is an eigenfunction, and since the potential $\tilde{q}$ is symmetric we conclude that $\tilde{v}_j = k_j\tilde{u}_j$. This shows that all $\tilde{y}_j$ vanish identically and the right-hand sides of equations (7), (8) and (9) are zero.

2. THE HOCHSTADT–HALD ALGORITHM

In this section we derive an algorithm for solving the discontinuous symmetric inverse Sturm–Liouville problem numerically. The idea of constructing an algorithm was originated by Hochstadt [2]. It was then refined and successfully implemented by Hald [1] in the continuous inverse Sturm–Liouville problem with a symmetric potential. We extend Hald's ideas to discontinuous inverse Sturm–Liouville problems with a symmetric potential. The results of numerical experiments using the new algorithm are given and errors in the examples are discussed.

The Hochstadt–Hald algorithm is based on the eigenvalue problems (1) and (2). The problem is to determine q and h of equation (1) when a, $\tilde{b}$, $\tilde{h}$, $\tilde{q}(x)$, $\{\lambda_j\}$ and $\{\tilde{\lambda}_j\}$ are given and $\lambda_j = \tilde{\lambda}_j$ for $j > n$. Here λ_j are the eigenvalues of equation (1) and $\tilde{\lambda}_j$ are the eigenvalues of equation (2). Note that only a finite number of eigenvalues differ. We use equation (9) to determine the relationship between q and the three terms $\tilde{q}$, $\tilde{y}_j$ and u_j. The denominator $\omega'(\lambda_j)$ can be computed using our knowledge of the eigenvalues λ_j, however a more computationally suitable method has been suggested by Hald. His ideas are presented below. To determine $\tilde{y}_j$ we solve the system (3)–(5). At $x = d$ and $x = \pi - d$ adjust u, u', w and w' using the jump conditions given in (2) and (3). We can now determine the boundary constant h using equation (7):

$$h = \tilde{h} + \frac{1}{2}\sum_{j=0}^{n} \tilde{y}_j(0). \tag{17}$$

To determine the potential $q(x)$ solve the system below for u_i:

$$u_i'' + \Big(\lambda_i - \tilde{q}(x) - \sum_{j=0}^{n} (\tilde{y}_j u_j)'\Big)u_i = 0 \tag{18}$$

$$u_i(0) = 1, \quad u_i'(0) = h \tag{19}$$

with discontinuities at $x = d_1$ and $x = d_2 = \pi - d_1$ satisfying the jump conditions

$$u(\tilde{d}_1+) = au'(\tilde{d}_1-), \quad u'(\tilde{d}_1+) = a^{-1}u'(\tilde{d}_1-) + \tilde{b}u(\tilde{d}_1-)$$
$$u(\tilde{d}_2-) = au(\tilde{d}_2+), \quad u'(\tilde{d}_2-) = a^{-1}u'(\tilde{d}_2+) - \tilde{b}u(\tilde{d}_2+)$$

for $i = 0, 1, \ldots n$. Here we use equations (7), (8) and (9) to determine h, b and q. The technique we have outlined yields the solution of the inverse Sturm–Liouville problem with symmetric potentials and symmetric discontinuities. In order to understand the construction of the algorithm, a brief discussion of the history of the problem must be given. Hochstadt [2] examined the continuous inverse Sturm–Liouville problem with two spectra. He constructed an algorithm based on a representation theorem in which $h = \tilde{h}$. Thus Hochstadt uses $\tilde{h}$ in equation (19) instead of h. Numerical investigations by Hald and later by the author show that a straightforward implementation of Hochstadt's algorithm to the continuous inverse Sturm–Liouville problem with symmetric potentials yields poor results; non-symmetric potentials and eigenfunctions which do not satisfy the right-hand boundary condition are found. And if the eigenvalues are sufficiently perturbed then the solution of (18) may go to infinity during the calculation. In [1] Hald modifies Hochstadt's algorithm by realizing that h cannot be equal to $\tilde{h}$ and setting h to be equal to $h + \frac{1}{2}\Sigma_{j=0}^{n} \tilde{y}_j(0)$. His changes are critical for the success of the algorithm. In the symmetric inverse Sturm–Liouville problem with jump discontinuities we follow the Hochstadt–Hald algorithm and set h equal to $\tilde{h} + \frac{1}{2}\Sigma_{j=0}^{n} \tilde{y}_j(0)$. In addition the constant b must be determined. Set $b = \tilde{b} + \frac{1}{2}(a^3 - a^{-1})\Sigma_{j=0}^{n} \tilde{y}_j(d-)u_j(d-)$ by the theorem in the previous section. That b is not equal to $\tilde{b}$ at the jumps is an analog of Hald's observation that h cannot equal $\tilde{h}$ at the boundaries. Thus our algorithm is a natural extension of the Hochstadt–Hald algorithm.

In order to give a precise and efficient algorithm to solve the Sturm–Liouville problem we note that $\tilde{v}_j(x) = \tilde{u}_j(\pi - x)$ for all x so that $\tilde{v}_j$ does not have to be calculated. In addition we must find a suitable method for calculating $\omega'(\lambda_j)$. From the Hadamard factorization theorem we have that

$$\frac{\omega(\lambda)}{\tilde{\omega}(\lambda)} = \frac{C}{\tilde{C}} \prod_{j=0}^{n} \frac{\lambda - \lambda_j}{\lambda - \tilde{\lambda}_j}. \tag{20}$$

Here we assume that the eigenvalues $\tilde{\lambda}_j$ are nonzero. We will show that the ratio $C/\tilde{C}$ equals one. Rewrite equation (20) as

$$\omega(\lambda) = \frac{C}{\tilde{C}} \prod_{j=0}^{n} \frac{\lambda - \lambda_j}{\lambda - \tilde{\lambda}_j} \cdot \tilde{\omega}(\lambda)$$

and differentiate. Then

$$\omega'(\lambda) = \frac{C}{\tilde{C}} \left(\prod_{j=0}^{n} \frac{\lambda - \lambda_j}{\lambda - \tilde{\lambda}_j} \right)' \cdot \tilde{\omega}(\lambda) + \frac{C}{\tilde{C}} \left(\prod_{j=0}^{n} \frac{\lambda - \lambda_j}{\lambda - \tilde{\lambda}_j} \right) \cdot \tilde{\omega}'(\lambda). \tag{21}$$

We now use that $\lambda_j = \tilde{\lambda}_j$ for all $j > n$ and consider the limit as $j \to \infty$. $\tilde{\omega}(\lambda_j) = 0$ and the term

$$\left(\prod_{j=0}^{n} \frac{\lambda - \lambda_j}{\lambda - \tilde{\lambda}_j} \right) \to 1$$

as $j \to \infty$ so that

$$\frac{\omega'(\lambda_j)}{\tilde{\omega}'(\lambda_j)} \to \frac{C}{\tilde{C}}$$

as $j \to \infty$. From a lemma in Kobayashi [3] we have that

$$\frac{\omega'(\lambda_j)}{\tilde{\omega}'(\lambda_j)} \to 1$$

as $j \to \infty$. Therefore $C/\tilde{C} = 1$. Now return to equation (21) to find that if λ_j is not an eigenvalue of equation (2) then

$$\omega'(\lambda_j) = \prod_{i \neq j} \frac{\lambda_j - \lambda_i}{\lambda_j - \tilde{\lambda}_i} \cdot \tilde{\omega}(\lambda_j). \tag{22}$$

Here we assume that λ_j is not an eigenvalue of (2). Let z_j be the eigenfunction of (2) corresponding to $\tilde{\lambda}_j$, and let w_j be the function $w_j = (\tilde{u}_j - z_j)/(\lambda_j - \tilde{\lambda}_j)$. Since $\tilde{\omega}(\lambda) = -\tilde{h}\tilde{u}(\pi) - \tilde{u}'(\pi)$ the last term in (22) is equal to $-\tilde{h}w_j(\pi) - w_j'(\pi)$ where w_j satisfies the differential equation

$$w_j'' + (\tilde{\lambda}_j - \tilde{q})w_j = -\tilde{u}_j$$

with boundary conditions

$$w_j(0) = w_j'(0) = 0$$

and jump conditions

$$w(d_1+) = aw(d_1-), \quad w'(d_1+) = a^{-1}w'(d_1-) + bw(d_1-)$$
$$w(d_2-) = aw(d_2+), \quad w'(d_2-) = a^{-1}w'(d_2+) - bw(d_2+).$$

See also section 1. If $\tilde{\lambda}_j \to \lambda_k$ with $k \neq j$ then we replace $\tilde{\lambda}_j$ and z_j in the above arguments by $\tilde{\lambda}_k$ and z_k.

An algorithm for solving the discontinuous, symmetric inverse Sturm–Liouville problem with symmetric potentials is given below.

Step 1°. For $j = 0, 1, \ldots n$ determine a k where $0 \leqslant k \leqslant n$ such that

$$|\lambda_j - \tilde{\lambda}_k| = \min_{i \in \{0, 1, \ldots, n\}} |\lambda_j - \tilde{\lambda}_i|.$$

Step 2°. For each $j = 0, 1, \ldots n$ solve the system given below on the intervals $0 < x < d_1$, $d_1 < x < d_2$ and $d_2 < x < \pi$

$$\begin{bmatrix} \tilde{u}_j \\ \tilde{u}'_j \\ w_j \\ w'_j \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \tilde{q} - \lambda_j & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & \tilde{q} - \tilde{\lambda}_k & 0 \end{bmatrix} \begin{bmatrix} \tilde{u}_j \\ \tilde{u}'_j \\ w_j \\ w'_j \end{bmatrix}$$

with the initial conditions

$$\begin{bmatrix} \tilde{u}_j \\ \tilde{u}'_j \\ w_j \\ w'_j \end{bmatrix}_{x=0} = \begin{bmatrix} 1 \\ \tilde{h} \\ 0 \\ 0 \end{bmatrix}$$

and the jump conditions

$$\begin{bmatrix} \tilde{u}_j \\ \tilde{u}'_j \\ w_j \\ w'_j \end{bmatrix}_{x=d_1+} = \begin{bmatrix} a & 0 & 0 & 0 \\ \tilde{b} & a^{-1} & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & \tilde{b} & a^{-1} \end{bmatrix} \begin{bmatrix} \tilde{u}_j \\ \tilde{u}'_j \\ w_j \\ w'_j \end{bmatrix}_{x=d_1-}$$

and

$$\begin{bmatrix} \tilde{u}_j \\ \tilde{u}'_j \\ w_j \\ w'_j \end{bmatrix}_{x=d_2+} = \begin{bmatrix} a^{-1} & 0 & 0 & 0 \\ \tilde{b} & a & 0 & 0 \\ 0 & 0 & a^{-1} & 0 \\ 0 & 0 & \tilde{b} & a \end{bmatrix} \begin{bmatrix} \tilde{u}_j \\ \tilde{u}'_j \\ w_j \\ w'_j \end{bmatrix}_{x=d_2-}.$$

Step 3°. For each $j = 0, 1, \ldots, n$ compute

$$\omega'(\lambda_j) = \frac{\Pi_{i \neq j}(\lambda_j - \lambda_i)}{\Pi_{i \neq k}(\lambda_j - \tilde{\lambda}_i)}(-\tilde{h}w_j(\pi) - w'_j(\pi)).$$

Step 4°. Set

$$h = \tilde{h} + \sum_{j=0}^{n} (\tilde{u}_j(\pi) - (-1)^j)/\omega'(\lambda_j).$$

Step 5°A. Solve the system given below on the intervals $0 < x < d_1-$, $d_1+ < x < d_2-$ and $d_2+ < x < \pi$

$$\begin{bmatrix} \tilde{y}_j \\ \tilde{y}'_j \\ u_j \\ u'_j \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \tilde{q} - \lambda_j & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \tilde{q} + \sum_{j=0}^{n} (\tilde{y}'_i u_i + \tilde{y}_i u'_i) - \lambda_j & 0 \end{bmatrix} \begin{bmatrix} \tilde{y}_j \\ \tilde{y}'_j \\ u_j \\ u'_j \end{bmatrix}$$

with the initial conditions

$$\begin{bmatrix} \tilde{y}_j \\ \tilde{y}'_j \\ u_j \\ u'_j \end{bmatrix}_{x=0} = \begin{bmatrix} 2(\tilde{u}_j(\pi) - (-1)^j)/\omega'(\lambda_j) \\ -2(\tilde{u}'_j(\pi) + (-1)^j/\tilde{h})/\omega'(\lambda_j) \\ 1 \\ h \end{bmatrix}$$

and the jump conditions

$$\begin{bmatrix} \tilde{y}_j \\ \tilde{y}'_j \\ u_j \\ u'_j \end{bmatrix}_{x=d_1+} = \begin{bmatrix} a & 0 & 0 & 0 \\ \tilde{b} & a^{-1} & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & b & a^{-1} \end{bmatrix} \begin{bmatrix} \tilde{y}_j \\ \tilde{y}'_j \\ u_j \\ u'_j \end{bmatrix}_{x=d_1-}$$

and

$$\begin{bmatrix} \tilde{y}_j \\ \tilde{y}'_j \\ u_j \\ u'_j \end{bmatrix}_{x=d_2+} = \begin{bmatrix} a^{-1} & 0 & 0 & 0 \\ \tilde{b} & a & 0 & 0 \\ 0 & 0 & a^{-1} & 0 \\ 0 & 0 & b & a \end{bmatrix} \begin{bmatrix} \tilde{y}_j \\ \tilde{y}'_j \\ u_j \\ u'_j \end{bmatrix}_{x=d_2-}$$

Step 5°B. At $x = d_1$ calculate

$$b = \tilde{b} + \tfrac{1}{2}(a^3 - a^{-1}) \sum_{j=0}^{n} \tilde{y}_j(d-)u_j(d-).$$

Step 6°. Set

$$q = \tilde{q} + \sum_{j=0}^{n} (\tilde{y}'_j u_j + \tilde{y}_j u'_j).$$

Note that Step 5°B takes place during Step 5°A since b must be calculated at $x = d_1$ before the jumps in u_j and u'_j are determined. In addition we remark that Step 6° takes place during Step 5°; q is evaluated as we solve for $\tilde{y}_j$, $\tilde{y}'_j$, u_j and u'_j. Finally note that $\tilde{y}_j$ is computed in Step 5°A even though it can be expressed in terms of the $\tilde{u}_j$ from Step 2° to avoid storing $\tilde{u}_j$ and $\tilde{u}'_j$ for all $j = 0, 1, \ldots n$.

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